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né en 1590 mort en 1665.

THE MATHEMATICS TEACHER

Volume XXIV Number 8

Edited by William David Reeve

Why Study Mathematics?

By F. L. Wren

George Peabody College for Teachers, Nashville, Tennessee

WHY SHOULD ANYONE study mathematics? This type of question is not peculiar to mathematics nor to the field of education. Let us look at a corresponding situation in the field of business. Suppose an automobile salesman attempts to sell a car, what are some of the questions he has to answer? The buyer wants to know the make of the car and compares it with other makes from the standpoint of beauty, service, and economy. Before the sale can be made the salesman must present convincing argument on all of these points and surely no real salesman will attempt such a task without being thoroughly familiar with the car himself. While the analogy may not be complete from the case of the automobile salesman to that of the teacher of mathematics, yet it is certainly true that the teachers of mathematics are primarily the ones who should be able to "sell" mathematics to the "doubting public." There are two questions that every mathematics teacher should be able to answer if he is to be able to give an intelligent answer to the one already proposed: they are *"What is mathematics?"* and *"What relation does mathematics have to the cultural, industrial, and recreational activities of a progressive civilization?"*

I. *What is Mathematics?* The meaning of mathematics has been food for thought to mathematicians and philosophers for many centuries, and to many people it has been, and still is, merely a con-

glomerated mass of signs, symbols, and multiplication tables. As a field of thought and endeavor the subject has at one time or another been recognized as a branch of physics, philosophy, and psychology, while Gauss is quoted as having referred to it as "the Queen of the Sciences."[1]

What then is mathematics? Is it, as Bertrand Russell once remarked, "The science in which we never know what we are talking about, nor whether what we are saying is true"?[2] Or is it, in the words of J. B. Shaw, "the sequoia that supports the universe of knowledge, deriving its stability from the roots that it sends out into the laws of nature, into the reasoning of men, into the accumulated learning of the dead? Whose trunk and branches have been built during the past ages out of the fibres of logic; whose foliage is in the atmosphere of abstraction; whose inflorescence is the outburst of the living imagination; from whose dizzy summit genius takes its flight; and in whose wealth of verdure its devotees find an everlasting holiday."[3] The first definition is probably too facetious on the surface to have any great significance until it is more thoughtfully analyzed, while the second, though metaphorically beautiful, is meaningless from the standpoint of a definition. Benjamin Peirce is responsible for the more explicit statement that "mathematics is the science which draws necessary conclusions from given premises."[4] The process of drawing necessary conclusions is fundamentally based upon a list of assumed premises and unproved propositions. These premises may be taken from experience or they may be any propositions sufficiently precise to make it possible to draw necessary conclusions from them. The conclusions are then no truer than the assumed premises and the real significance of Russell's statement becomes more evident. Keyser says, in discussing Russell's statement, that "a juster *mot* would be: Sheer mathematics is the science in which one never thinks of a definite sort of subject-matter nor fails to know that what one asserts is true."[5]

[1] Shaw, J. B.—*The Philosophy of Mathematics*. Open Court Publishing Co., 1918, pp. 4-5.

[2] Russell, Bertrand—*International Monthly*, 4, 1901, p. 84.

[3] Shaw, J. B.—*loc. cit.*

[4] Peirce, Benjamin—*Linear Associative Algebra*. Section 1. Lithographed, 1870. Reprinted in the *American Journal of Mathematics*, 1881, 4, 97.

[5] Keyser, C. J.—*The Pastures of Wonder*. Columbia University Press, 1929, p. 77.

Mario Pieri is responsible for the statement that "mathematics is an hypothetico-deductive system." This simply means that mathematics is a system of logical processes whereby conclusions are deduced from whatever fundamental assumptions there may be hypothesized. To think mathematically does not necessarily mean that the individual is thinking in signs, symbols, and multiplication tables. Mathematical thinking can be done in any situation, the signs and symbols are merely the shorthand to free the situation from any superfluous complexities. In other words, to think mathematically is to free oneself from any peculiarity of subject-matter and to make inferences and deductions justified by the fundamental premises; to think mathematically is to develop and carefully weigh, one against the other, the various differential characteristics of the interrelations of objects of the physical or mental world and then to deduce from them the truths which they imply.

One definition of science is "accumulated and accepted knowledge systematized and formulated with reference to the discovery of general truths or the operation of general laws." Mathematics, then, is a science in which the discovery of general truths is made by inference or deduction from previously assumed or established truths, all of which are based upon certain fundamental assumptions and definitions. It is not a substitute for experimental science but a helpmeet in the sense that it can take the results of experiments and observations, systematize them, and thus differentiate between what is fundamental principle and what is not. Mathematics is the science of implications and inferences.

II. *What relation does mathematics have to the cultural, industrial, and recreational activities of a progressive civilization?* What is the relation of the science of mathematics to the rather varied and complex activities of a progressive civilization? This relation of mathematics to civilized progress can best be appraised through an evaluation of its contributions to and influence upon the life of the individuals whose efforts and ideals have brought about this progress. Much has been said about the utilitarian values of mathematics and these cannot be unduly emphasized if we remember that there are other values just as important. It would seem that a better appraisal could be obtained by analyzing the values to be derived from mathematics into the practical, historical, mental, moral, religious, esthetic, and recreational values.

1. *The practical values.* The practical values of mathematics are on exhibit in almost every phase of life. The full significance of this statement becomes more evident when an attempt is made to answer the question: "What would happen if all the influence of mathematics and mathematical research were cut off from the life about us?" The radio, the wireless telephone and telegraph, which are the direct results of the mathematical and physical calculations of Maxwell and Hertz, would no longer be ours to use. The structure of every bridge and building would be a hazard to life in general since their safety is dependent upon the mathematical calculation of strains and stresses. The industrial, financial, and engineering worlds would no longer be able to operate with their characteristic precision and system. All scientific experiment would be seriously impaired if not entirely impossible.

One would hardly raise the question as to the value of mathematics in the experimental sciences of physics and chemistry. There are studies which justify the assertion that, even on an elementary level, mathematics is essential to a proper approach to and appreciation of these sciences. These studies deal only with the elementary phases of the sciences and the mathematics involved expands with the frontiers of the science, in fact, mathematics with its symbols and formulae quite often leads the way. The mathematical investigation expresses, in its formulae, truths which are later verified by experiment. This has been the history of physical experiment in particular and today the world's imagination has been stirred by the mathematical calculations of Einstein and the possibilities of his new field theory. Just as in physics and chemistry, mathematics has been a frontiersman in the field of astronomy, and the story of its contributions to this field of scientific investigation is an epic of the ages.

It is only of recent years that biologists, in general, have begun to realize the vast possibilities growing out of the application of the technique of mathematics to their science, and now it is quite commonly agreed that "the possible contributions of mathematics to biological science are too varied to be succinctly summarized." Quetelet, Galton, and Pearson were the frontiersmen who broke the barriers and now they are followed by many who say that biological experiment must be submitted to mathematical treatment for full interpretations.

We are not limited to the exact sciences alone for such examples of the use of mathematics; sooner or later every true science tends

to become mathematical. The applications of statistics to psychology, education, and the various phases of the social sciences is but a form of mathematical technique. Many economic principles have been established as mathematical theorems and it has been said that the economic world is a world of *n* dimensions.

The field of agricultural mathematics is gradually becoming one of importance. We are told that the field of agriculture, in its various ramifications, calls for a knowledge of mathematics ranging from the fundamentals of arithmetic and algebra through the theory of depreciation and compound interest to applications of the calculus and the elementary theory of errors. When one makes the statement that a Ph.D. thesis of fifteen years ago entitled "A Study of the Plow Bottom and Its Action Upon the Furrow Slice" has had profound influence on modern plow design and usage, there is no occasion for much surprise, but when the statement is accompanied by the information that this is "a scholarly treatise of thirty-four pages, twenty-four of which are pure mathematics of a high and difficult order," there is something of the unusual and unexpected in it.

A certain amount of mathematics is useful in pharmacy, dentistry, and nursing. The problem of ore location leads to mathematical problems of a very difficult type. The same thing is true in the firing of long range guns, the transmission of messages over telephone wires, the transmission of electrical currents, and the solving of problems arising from submarine cable telegraphy.

It would be possible to continue indefinitely listing examples of the practical applications of mathematical theory and technique, but, in the words of Professor Hedrick, "What does it profit a man if he learn every fact and acquire every skill of mathematics, if he loses the soul of the subject?"

2. *The historical values.* The historical values of mathematics, while less tangible than the practical values, are no less significant. The story of civilization is rich in its experiences of trying and erring, groping and stumbling. Human knowledge has developed in this way. The history of the development of mathematical knowledge has been just such a story, crowded with the names of great leaders of thought, inventors, discoverers, architects, musicians, poets, philosophers, men and women in general who have been among the leaders in the struggle toward a civilized world.

There is evidence of a prehistoric notion of number just as there is evidence of a prehistoric man, but what this number concept was

is more or less speculation. This primitive idea of number probably consisted merely of counting, and as this number sense developed there evolved a rather uniform number language. The history of counting up to the time of the invention of the principle of position is rather characterized by paucity of achievement. This covers a period of many centuries which saw the rise and fall of many a civilization whose progress was considerably impaired by its crude system of numeration. Written numeration of some form or another has possibly been in existence ever since there was any private property, and it is not very difficult to conceive of how crude and inflexible it must have been without the aid of the principle of position, which carries with it a symbol for an empty column. The notion of zero is somewhat commonplace with us today although it still causes trouble at times in our calculations. It is such an integral part of our number system that it is hard to conceive of the many centuries of mankind that existed with no knowledge of its significance or use. The discovery of the principle of position and the invention of zero have been heralded down the ages as world-events of no little importance.

The story of zero is but one of the many fascinating portions of mathematical history. There is the same struggle for existence in the history of negative numbers, irrational numbers, and complex numbers, enhanced somewhat by a thread of superstition and unbelief running throughout the story. How the calculus, born of the research of ancient times on the "three famous problems," nourished on the Method of Exhaustions of the Greeks and the Method of Indivisibles of Cavalieri, developed into the Infinitesimal Calculus of Leibniz and Newton, is another of the great stories found in mathematical history. These and many other just such human element stories make the history of mathematics a valuable contribution to the history of civilization.

3. *The mental values.* Surely any subject that has played such a vital part in the intellectual development of the human race has mental values which deserve consideration and recognition. The abstractions of algebra, the formal logic of a geometrical demonstration, the induction and deduction, the synthesis and analysis that characterize mathematical thought give mental training that can be found in no other field of endeavor.

Dewey[6] has defined reflective thought as "active, persistent, and

[6] Dewey, John—*How We Think*. D. C. Heath & Co., 1910, p. 6.

careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it, and the further conclusions to which it tends." In the light of this definition, any act of reflective thinking becomes a process of forming "hypothetico-deductive" judgments, that is, accepting certain assumed or established propositions and definitions and making whatever deductions and implications they justify. Rignano[7] refers to mathematical reasoning as the highest form of the logical process. The characteristics of mathematical thinking are (1) the ability to set up clear-cut premises and definitions, (2) the ability to reason coherently and critically, and (3) the ability to draw implied conclusions. These three characteristics are present whether the mathematical thought has to do with abstract generalities or concrete actualities. Mathematical training then becomes a sort of refining process whereby the wires of "the thinking machine" become more highly sensitive and more delicately selective and the "machine" more acutely attune to producing harmony out of the chaotic ether of human progress.

4. *The moral values.* A subject has moral values in so far as it helps one to discriminate between right and wrong, or between good and bad conduct, and in so far as it helps him to appreciate his place in the scheme of things. If the human race were a race of robots so mechanically perfect that each individual always did the right thing at the right time, then there would never arise the question of moral conduct. Or, if it were a race of individuals, no one of which had the mental capacity to make a decision between right and wrong, or between good and bad conduct, there would be no moral responsibility. Moral conduct and responsibility enter into an individual's life at the moment when he faces an alternative between what is right and what is wrong and has to make a choice for himself. The fundamental pattern of mathematical thought is that of making reflective judgments, of thinking impartially and systematically. These are the values that make for moral conduct.

Man to be moral must appreciate his place in the scheme of things, his proper relation to man. Where can he better learn this than through the medium of mathematics to come to some realization of the mathematical exactness in the order of the universe about him, and also to some realization of the vast expanse of this universe in time and space?

[7] Rignano, E.—*The Psychology of Reasoning*. Harcourt, Brace and Co., 1927.

5. *The religious values.* The recognition by man of his infinitesimal nature in the universe around him, and the realization that there is a "harmony of the spheres" but help him to acclaim with Voltaire that "all nature cries aloud that God does exist; that there is a supreme intelligence, an admirable order, and that everything teaches us our own dependence upon it." Reflection upon the laws governing finite quantities and how they no longer are necessarily true in the realm of the infinite helps to rationalize the existence of a "Man of God" endowed with powers which are unencumbered by human limitations.

Any system of constructive thought is no stronger than its fundamental assumptions and definitions. If these are not accepted the whole philosophic scheme breaks down. Lobatschewskian and Riemannian geometries are just as logically sound as our commonly accepted Euclidean geometry. There is no more reason, logically, why we should accept the fundamental postulates and definitions of one of these over those of another, but we accept the Euclidean postulates because they seem to conform more nearly with our every-day experience. Having accepted our fundamental assumptions we know that whatever truths they imply are eternal, universal, and unchangeable. These truths do not have to be verified experimentally, but are accepted without question so long as we hold to the fundamental assumptions and definitions.

Religious creeds differ in their fundamental assumptions and beliefs. Complete acceptance by an individual of any particular creed means the acceptance of these fundamental assumptions and beliefs and the willingness to interpret life in the light of the truths they imply. Mathematics points the way to accepting these truths as being eternal, universal, and unchangeable, and further shows to man "the futility of setting up his childish arrogance of disbelief in that which he cannot see."

6. *The esthetic values.* The contemplation of the pattern of our universe impresses us with the truth of the statement made by that philosopher of old that "God eternally geometrizes." This is evidenced by the symmetry of the human form and of the structure of plant life, and in the geometric forms found in crystals, in vegetation, and in animal life. Hambridge[8] says that there are two kinds of symmetry in art, namely, static and dynamic. Static symmetry is

[8] Harbridge, J.—*Dynamic Symmetry.* The Greek Vase. Yale University Press. 1920; *The Elements of Dynamic Symmetry.* Brentano's, Inc., 1926.

that symmetry which is characterized by a passiveness and which every artist uses more or less unconsciously. Dynamic symmetry is characterized by a certain activeness such as the symmetry of man and plants. The ordinary garden variety of the sunflower, according to Hambridge, exhibits some rather interesting ideas of what is meant here by dynamic symmetry. Its fruits are enclosed in rhomboidal sockets whose walls trace out intersecting curves which closely resemble logarithmic spirals.

Someone has said that "music is number made audible while architecture is number made visible." The relation of mathematics to music goes back to Pythagoras who "is said to have discoverd that the fifth and the octave of a note can be produced on the same string by stopping at two-thirds and one-half of its length, respectively, and it is thought that this harmony gave rise to the name of 'harmonic progression.'"[9]

In architecture geometrical forms and the principles of symmetry assume very fundamental rôles. The play of the imagination is the source of ingenuity in the creation of new designs and plans and the introduction of notions of dimensionality is but one way that mathematics has of giving free reign to the imagination. The possibilities for creative architectural design are enhanced by the conception of the fourth dimension and the magic square.[10]

7. *The recreational values.* The proper appreciation of beauty of form is a contribution that mathematics makes toward the proper use of leisure time. The expansion of man's field of thought to include a familiarity with the technique of mathematics provides him with a means of keeping in touch with a certain amount of scientific progress. For the mathematician who is interested in the subject for its own sake there is the joy and beauty of mathematical creation just as for the musician there is the joy of musical composition.

When looked at purely from the standpoint of play, one must realize that number is fundamental to almost all games in varying degrees of complexity, and there are many amusing mathematical recreations that can afford many moments of interesting and entertaining diversion.

8. *The relation of mathematics to a progressive civilization.* The cultural activities of a progressive civilization are those activities which

[9] Smith, D. E.—*History of Mathematics.* Ginn & Co., 1925, II, p. 75.

[10] Bragdon, C.—*Architecture and Demoracy.* Alfred A. Knopf, New York., 1926.

are characteristic of the mental and moral attainments of a people so disciplined by its contact with the sum total of human knowledge that these activities represent progress and not retrogression. Whether there is progress or not is determined by the history of the development of the human race. The influences of mathematics and mathematical thought are indelibly imprinted upon the pages of human progress. The history of mathematics is rich with the poetic beauty of discovery and invention; the processes of mathematical thought are invaluable to the mental and moral enlightenment of a social order; and the methods of mathematical design are fundamental to art and architecture.

The industrial activities of a progressive civilization are those activities which are directed toward producing better conditions governing the employment of labor and capital. Mathematical research has long been influential in directing the development of more efficient machinery and better housing conditions. It has pointed the way to faster and safer transportation, and to vast possibilities in the conservation of time, space, and energy. Of recent years mathematical thought has more and more entered into the discussion of the problems of the social sciences. If all of the influence of mathematics and mathematical research were removed the industrial world would be hopelessly paralyzed.

The recreational activities of a progressive civilization are those activities which better enable the individuals of a social order to more beneficially employ their leisure time. Such activities are directed along the lines of intellectual, moral, and religious pursuits as well as along the lines of play. Mathematical training widens the horizon of intellectual endeavor, rationalizes moral conduct and responsibility, justifies the pursuit of religious ideals, and enriches the field of play.

III. *Why study mathematics?* "True education, always personal, will develop the social consciousness and promote genuine social culture."[11] The educative process, then, is a process whereby an individual acquires those experiences which will better enable him to function as a member of his social order. We have seen how mathematics is inextricably woven into the educational texture across the warp of civilized progress. The study of mathematical subject matter and technique prepares an individual for better adjustment to a progressive environment and for more efficient functioning as a member of a civilized social order.

[11] Stuckenberg, J. W. H.—*Sociology*. New York, 1903, II, p. 272.

The Problem of the Teaching of Exponents

By William James Lyons (*St. Louis University*)
St. Louis Missouri

I. Preview

Though the problems arising in the teaching of exponents cannot be said to be insignificant, there has been little written on the subject, as compared to the material on the teaching of other topics in the algebra course. We will not here seek to define a reason for the dearth of material on the teaching of exponents. Suffice it to say that the task of teaching exponents so that they will be meaningful and understandable to the pupils presents difficulties. The writer has collected suggestions from articles and works on the teaching of secondary mathematics, and has analyzed the treatment of exponents in several textbooks. In this paper he has endeavored to choose and refine these various suggestions so as to produce a single, unified method of teaching exponents.

II. Objectives of Teaching Exponents

It will be of value and help to the teacher of algebra in laying out his unit on exponents, and as a standard by which to evaluate technique, to be familiar with the objectives and requirements in teaching exponents. Reeve lists as such objectives: "To develop the following abilities:

"1. To know the definitions of negative, zero, and fractional exponents.

"2. To know and to use the law $a^m \cdot a^n = a^{m+n}$.

"3. To know and to use the law $a^m/a^n = a^{m-n}$.

"4. To know and to use the law $(a^m)^n = a^{mn}$.

"5. To know and to use the law $(abc)^m = a^m b^m c^m$.

"6. To know and to use the law $(a/b)^m = a^m/b^m$.

"7. To prove the law $a^m \times a^n = a^{m+n}$ for positive, integral exponents.

"8. To prove the law $(a^m/a^n) = a^{m-n}$, m being greater than n, for positive, integral exponents.

"9. To prove the law $(a^m)^n = a^{mn}$ for positive, integral exponents.

"10. To perform the fundamental operations with expressions containing positive, zero, and negative exponents."[1]

Reeve also includes, under his list for radicals, the following: "To transform an expression which contains fractional exponents to one which contains radicals, and to reverse the transformation."

It should be noted that this list is for intermediate, or advanced secondary algebra. Accordingly, several of the items in the above list would not appear in lists for elementary algebra. In fact there are textbooks that omit treatment of either zero, negative, or fractional exponents, or several of these. Such textbooks are usually meant to be used only during the first year. There are undoubtedly teachers and writers, though, who would apply the above list of objectives *in toto* to elementary algebra. But there is some evidence for concluding that Items 7 to 9 do not apply to the first course. We are told that the essentials of algebraic technique should include "the laws for positive integral exponents, and the meaning and use of fractional and negative exponents, but not the formal theory."[2] (We call attention to the last phrase.) Of course, it is doubtful just how much understanding of theory is implied by the term *meaning* as here used. On the other hand, we are told, in a first course in algebra "the definitions of negative, zero and fractional exponents should be given and it should be made clear that these definitions must be adopted if we wish such exponents to conform to the laws for positive integral exponents. Reduction of radical expressions to those involving fractional exponents should be given, as well as the inverse transformation."[3]

III. Definition of "Exponent"

An effort should be made by the instructor to make his introduction to exponents a definition as well. This suggestion, of course, has wider application than to merely exponents, or even mathemathics.

[1] W. D. Reeve, "Objectives in Teaching Intermediate Algebra," THE MATHEMATICS TEACHER, XX (March, 1927), 156–57.

[2] The National Committee on Mathematical Requirements, "College Entrance Requirements in Mathematics," THE MATHEMATICS TEACHER, XIV (May, 1921), 232.

[3] The National Committee on Mathematical Requirements, "Elective Courses in Mathematics for Secondary Schools," THE MATHEMATICS TEACHER, XIV (April, 1921), 164–65.

The teacher might recall to the pupils that to show that a number is multiplied by itself we write $x \times x$, or xx. If we wish to multiply again by x we write xxx. It should then be pointed out that this practice is rather cumbersome and inconvenient, especially if a quantity is multiplied by itself several times. To simplify his work Descartes, in the 17th century, wrote x^2 for xx, x^3 for xxx, and so on. The exact position of the 2 and 3 should be dwelt on, and their name *exponents* here introduced. In recapitulation: An exponent tells how many times the letter x would appear as a factor if the quantity were written out in full.

Following this several literal and numerical examples may be presented, such as:

$$x^5 = x \cdot x \cdot x \cdot x \cdot x,$$

$$x^9 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x,$$

and if $x = 3$,

$$x^2 = x \cdot x = 3 \times 3 = 9.$$

Pupils are apt to adopt a more sympathetic attitude (and hence one favorable to learning) if the teacher stresses the point that x^5 is simply a more convenient and economical way of writing $x \cdot x \cdot x \cdot x \cdot x$. Many of the textbooks examined by the writer formally define an exponent, and then mention parenthetically that a^2 means $a \cdot a$. Such procedure is obviously undesirable.

In an introduction to exponents, it would be well for the teacher to point out the distinction between *exponent* and *coefficient*. One good introduction is to define such terms as *base*, *exponent*, and *power* by specific, numerical examples, but it should be followed by a discussion such as is outlined above. The definition by numerical example is very good, perhaps, for getting the terms clearly in the pupil's mind; but with this demonstration alone, it is highly probable that the pupil does not fully grasp the idea that the base is a repeated factor—an important objective.

IV. The Theory of Exponents

The number of theorems relating to exponents, and stated as such, varies from two to fifteen with different writers. Some writers consider to be laws what others treat as applications or special cases. Rules are useful when they serve as guides, and are under-

stood. In the teaching of exponents we undoubtedly have need for more than two rules if we are to clarify the high points of the unit in the pupil's mind. Conversely, we must restrict the number of laws if we are to avoid confusion and wasteful application. Careful analysis shows that not more than eight or nine statements deserve the emphasis of theorems, or laws. It may be maintained that there is no harm in presenting several laws, even though some be unimportant, or that a certain amount of study is demanded by the subject, and this may as well be applied to additional theorems as to specific problems. The psychological fallacy here is that with the whole topic becoming a sequence of nothing but laws, the important laws lose their value as guides. The pupil does not distinguish between the Law of Multiplication and the special law for the multiplication of bases to fractional powers, with reference to their importance. The general objectives of mathematics teaching dictates that the maximum number of particular cases be brought under the minimum number of laws or theorems.

The first law to be presented is the so-called Law of Multiplication:

$$x^a \cdot x^b = x^{a+b}.$$

But this law should not be presented in this form. Pupils of algebra at this point are familiar with numerical exponents (though they cannot operate on them) but cannot comprehend literal exponents. The best, and probably the usual practice is to present such a special case as

$$x^3 \times x^4 = x^{3+4} = x^7.$$

It should be pointed out that

$$x^3 \times x^4 = x \cdot x \cdot x \times x \cdot x \cdot x \cdot x = x^7.$$

Several such cases should be given, and attention called to the fact that in every example the exponent of the product is the sum of the exponents of the multipliers, or factors. This is not a proof, but it is an effective teaching technique. Whether the teacher may stop with that development depends upon the ability of the class, and upon the objectives of the course.

In more advanced classes it might be pointed out that

$$x^a = x \cdot x \cdot x \text{ to } a \text{ times},$$

and

$$x^b = x \cdot x \cdot x \text{ to } b \text{ times},$$

so that

$$\begin{aligned} x^a \cdot x^b &= x \cdot x \cdot x \text{ (to } a \text{ times)} \times x \cdot x \cdot x \text{ (to } b \text{ times)}, \\ &= x \cdot x \cdot x \text{ (to } a+b \text{ times)}, \\ &= x^{a+b}. \end{aligned}$$

Followed through in these steps the formal theory of the Multiplication Law should be comprehensible to most advanced algebra students.

The second, the Law of Division,

$$x^a \div x^b = x^{a-b}$$

is to be taught with the same technique as the first law. But it is advisable that this precaution be taken: in numerical examples always take $a > b$. This will eliminate the confusion which will arise in the pupils' minds with the appearance of negative exponents. Since this topic is followed closely by a treatment of negative exponents, there is but little danger that pupils will set up a mental frame which ignores the existence of negative exponents, and would make difficult their presentation later.

The relation most generally taught as the third is expressed by the formula

$$(x^a)^b = x^{ab}.$$

Again, it would be well to present this first with numerical exponents:

$$\begin{aligned} (x^2)^3 &= (x^2) \cdot (x^2) \cdot (x^2), \\ &= x^2 \cdot x^2 \cdot x^2, \\ &= x \cdot x \times x \cdot x \times x \cdot x = x^6. \end{aligned}$$

After the class has been taken through several such special developments, the general formula and its proof may be presented. The syllabus may require that this be presented in a later course. In either case the following is a good demonstration:

$$(x^a)^b = x^a \cdot x^a \cdot x^a \text{ (to } b \text{ times)}.$$

That is, we will have a certain number, b, of (x^a)'s, multiplied together. From the Multiplication Law we have, then,

$$\begin{aligned} (x^a)^b &= x^{a+a+a \cdots \text{(to } b \text{ times)}}, \\ &= x^{ab}. \end{aligned}$$

The teaching of fractional and negative exponents can be facilitated if the teacher has an understanding of the nature of exponential laws. The element of convention in exponents is very probably unrecognized by many teachers.

From the original definition of an exponent the meaning of fractional and negative exponents does not *logically* follow. Rather, these latter become meaningless, since we said only that exponents tell the number of times the base is taken as a factor; such a definition excludes fractional and negative numbers as exponents. From the original definition it is impossible to prove that $x^{1/2}=\sqrt[2]{x}$. It is necessary to define specifically negative and fractional exponents, to adopt some convention with respect to them. For convenience, we so define these kinds of exponents that the same basic laws apply to them as apply to positive, integral exponents. Hence we arrive at the definitions or laws which follow. Obviously, it is not meant that the material of this paragraph should be the subject-matter for classroom discussion in a high school algebra class.

An exposition of the meaning of zero exponents may well precede or introduce the subject of fractional exponents. Some textbooks present zero exponents at the end of the unit on exponents. I have here followed the order of other writers. As in the previous demonstrations the teacher should first present several numerical cases. To evaluate $x^3 \div x^3$ the law of division should be recalled to the pupils:

$$x^3 \div x^3 = x^{3-3} = x^0.$$

It should then be pointed out to them that by the law (with which they are already familiar) that quantities divided by themselves equal unity or 1, we have

$$x^3 \div x^3 = 1.$$

The only probable difficulty in this step will be in getting the pupils to see x^3 as a single, concise quantity. The presence of the exponents may at first hinder such an understanding. The next step is that, since things equal to the same thing are equal to each other,

$$x^0 = 1.$$

The general rule may then be presented.

There is a common tendency for the child-mind to regard $x^0=0$. The child apparently reasons that since x^2 means that "x is taken

twice," and x^3 that "x is taken three times," x^0 means that "x is not taken at all," that it is not brought into the problem, and hence may be replaced by zero. An emphasis on the understanding of the general rule, and drill seem to be the only means of meeting this problem.

To the algebra pupil $x^{1/2}$ or $x^{1/3}$ are inexplicable, meaningless. He cannot conceive of a quantity being *raised* to a power *less* than that quantity already has. The only probable solution he sees is $x^{1/2}=x/2$. In developing a meaning for fractional exponents the teacher should refer the class to the rules which the pupils have seen exponents obey. It is not necessary, or advisable for the teacher to discuss the logic of applying these laws for integral exponents to fractional exponents.

According to these laws,

$$x^{1/2}\cdot x^{1/2} = x^{1/2+1/2} = x^1, \text{ or } x.$$

Then, since $x^{1/2}$ multiplied by itself gives x, $x^{1/2}$ must be another way of writing the square root of x. The pupils already know that the square root of a quantity is that part which, multiplied by itself gives the original quantity. Hence,

$$x^{1/2} = \sqrt{x}.$$

Similarly, the meanings of $x^{1/3}$, $x^{1/4}$, etc., may be developed. The conclusion to be drawn may then be formulated, and presented:

$$x^{1/n} = \sqrt[n]{x}.$$

To point out the meaning of $x^{3/2}$ the teacher should first decompose the exponent:

$$x^{3/2} = x^{1/2+1/2+1/2}.$$

By the previous rules,

$$x^{1/2+1/2+1/2} = x^{1/2}\cdot x^{1/2}\cdot x^{1/2} = (x^{1/2})^3 = (\sqrt{x})^3.$$

That is, $x^{3/2}$ means the cube of the square root of x. It should be shown, too, that

$$x^{3/2}\cdot x^{3/2} = x^3,$$

so that

$$x^{3/2} = \sqrt[2]{x^3}.$$

Cases of other improper fractions might also be brought before the class:

$$x^{4/3} = x^{1/3+1/3+1/3+1/3} = x^{1/3}\cdot x^{1/3}\cdot x^{1/3}\cdot x^{1/3},$$
$$= (x^{1/3})^4 = (\sqrt[3]{x})^4.$$

The general rule for fractional exponents may then be formulated:

$$x^{a/b} = (\sqrt[b]{x})^a = \sqrt[b]{x^a}.$$

The meaning of negative exponents should be brought out by referring to the law of division:

$$x^3 \div x^4 = x^{3-4} = x^{-1}.$$

But

$$\frac{x^3}{x^4} = \frac{\cancel{x}\cancel{x}\cancel{x}}{\cancel{x}\cancel{x}\cancel{x}x} = \frac{1}{x}.$$

So that, since the left-hand members of these two equations are identical,

$$x^{-1} = \frac{1}{x}\cdot$$

The meanings of x^{-2}, x^{-3}, etc. may similarly be derived. Pointing out the general rule

$$x^{-a} = \frac{1}{x^a},$$

should complete the demonstration work on negative exponents.

V. Summary —Pupil-Understanding of Exponents

With respect to the understanding by the pupils of such laws as $(x^a)^b = x^{ab}$, $x^{a/b} = \sqrt[b]{x^a}$, and $x^{-a} = 1/x^a$, it is doubtful whether we should require more than their recognition as empirical laws, inductively arrived at. (I do not here refer to formal *mathematical induction.*) It would appear to be sufficient if the pupil can show that $x^{3/2} = \sqrt{x^3}$, and can state the general rule. The immaturity of the average high school pupil would seem to make impossible a genuine comprehension of a rigorous, general proof of these laws. On the other hand, general proofs of the first two laws are probably manipulable by high school students. We cannot agree with those educators who apparently are content if pupils are able only to remember, and use with facility the rules of exponents.

Not only should we show pupils these laws, but should expect also a degree of assimilation of their derivation. It is true that teachers and mathematicians apply principles mechanically. But they can afford to take short-cuts, since they have mastered the theory (or should have), and can refer to it when necessary. Memory of laws and facility in operation are to be desired, but some understanding of principles is in addition necessary for flexibility in application of laws. That this is the point of view of most authorities may be seen from the discussions of the objectives of mathematics.

A common occurrence in a secondary algebra class is for a student who knows that $n^3 \times n^4 = n^7$, to say that $3^3 \times 3^4 = 9^7$. Such an error is symptomatic of memoriter learning of exponential laws. The pupil here mechanically applies the multiplication law to the exponents, and imposes upon the result an operation carried over from his arithmetic training. Such errors are a signal to the teacher that the pupil needs reteaching on the nature of exponents.

If $a+b+c=0$

THE MATHEMATICS TEACHER recently had a letter from Mr. Hermann T. R. Aude of Colgate University at Hamilton, New York, in which he said:

At Colgate we are interested in your journal and are using it in connection with our "Teachers Course." Our students find a great deal of interest and profit in it.

(1) Mr. Aude then enclosed the following:

Suppose three numbers are selected so that their sum is zero. For an example choose 3, 4, and —7. With these three numbers as coefficients it is possible to form six different quadratic equations:

$$3x^2+4x-7=0, \qquad 4x^2+3x-7=0,$$
$$3x^2-7x+4=0, \qquad -7x^2+4x-3=0,$$
$$4x^2-7x+3=0, \qquad -7x^2+3x+4=0.$$

The interesting point is that every one of these equations has rational roots. This will also be true for those equations formed from the set 2, —5, 3. In fact, it is true for any set of three rational numbers a, b, c, provided their sum is zero.

The proof is simple. It will without doubt interest a class in intermediate algebra to work it out. It is therefore left for those students and teachers who are interested.

Sources of Program Material and Some Types of Program Work Which Might Be Undertaken by High School Mathematics Clubs

By RUTH HOAG

Natrona County High School, Casper, Wyoming

THE FOLLOWING SUGGESTIONS for work of high school mathematics clubs are listed under seven headings: (1) Projects; (2) Demonstrations and Experiments; (3) Skits, Plays, and Dialogues; (4) Games; (5) Field Trips and Field Work; (6) Outside Speakers; and (7) Topics for Group or Individual Presentation. A project may be initiated and carried forward in addition to a diversified or unified program meeting if desired. Such projects can aid in allowing the club work to parallel and grow out of curricular work.

The numbers which appear beneath each topic refer to the bibliography of mathematical references which is placed at the close of this paper. The number of the reference is followed in some instances by the first page number of specifically related material. These references are by no means exhaustive and many other contributions to the given topics will be found in the books and references of the bibliography as well as elsewhere. An example of the use of this material in the preparation of a given topic follows:

Topic: The Principle of Duality (114 p. 57) 114 refers to Young, J. W. A.; *Monographs on Topics of Modern Mathematics;* which is preceded by the number 114 in the following bibliography. The source material begins with page 57.

PROJECTS

1. Compilation of a dictionary of terms and symbols used in high school mathematics. These can be collected from high school texts and should be worded and explained concisely. A number of texts in the various subjects would need to be reviewed.

2. A Mathematical scrap book. This may be undertaken by each individual for his own use, or be prepared by the club to be placed finally in the school library. Club meetings can be used in its preparation if the work is carefully planned in advance of each meeting.

3. The construction of instruments to be used later in field work. A list of necessary instruments, plan for constructing them, and ma-

terials needed will be considered before the actual work of construction can begin. (81, pp. 84-88).

4. *The writing of a history of mathematics for high school students.* The club meeting may consist of round table discussion and actual work on the project if the work is carefully outlined and properly apportioned. (4, 3, 6, 12, 10, 9, 18, 17, 13, 24, 27, 25, 34,35, 39, 41, 46, 45, 47, 52, 48, 63, 75, 77, 104, 105,95, 87, 100, 102, 106, 108, 112, 114, 113, 103).

5. *The construction of a set of models to be used in the solid geometry classes.*

6. *The preparation of a mathematics paper.* This sheet should contain club news, news concerning mathematical work of the school and short articles of a mathematical nature written by students. Some suitable names for such a paper are: "The Euclidean," or "The Parapegma." Writing the name in Greek letters adds interest to the project. Stencils can be cut by club members, and the page mimeographed.

DEMONSTRATIONS AND EXPERIMENTS

1. Soap bubbles and mathematics. The demonstration of the sphere, ellipsoid, paraboloid, and hyperboloid through the use of soap, water, pipes, and glass sections.
2. The finding of pi by chance. (36; 110 p. 127).
3. Paper folding and plane geometry. (86).
4. Slide rule demonstration. (22; 87 p. 343).
5. The steel square. (81, pp. 282-285).

SKITS, PLAYS, AND DIALOGUES

Some of the best and most applicable skits and dialogues will be written by members of the club. A brief list of some of these types of activity follows:

1. "Alice in the Wonderland of Mathematics," a story which might be dramatized. (110, p. 218).
2. "A Near Tragedy," by Florence Brooks Miller (THE MATHEMATICS TEACHER; Oct. 1928).
3. "Evolution of Numbers, An Historical Drama in Two Acts," by H. Slaught (THE MATHEMATICS TEACHER; Oct. 1928).
4. "Falling in Love with Plain Geometry," by Caroline Hatton and Doris Smith (THE MATHEMATICS TEACHER; Nov. 1927).

5. "Geometry Humanized," by Erma Scott (The Mathematics Teacher; Feb. 1928).

6. "If," by Ruth L. Snyder (The Mathematics Teacher; Dec. 1929).

7. "Little Journeys to the Land of Mathematics," by Alma Crawford (The Mathematics Teacher; Oct. 1924).

8. "A Mathematical Nightmare," by J. Skerrett (The Mathematics Teacher; Nov. 1929).

9. "Mathesis," by Ella Browell (The Mathematics Teacher; Dec. 1927).

10. "Number Play in Three Acts," by T. Schlierholz (The Mathematics Teacher; Mar. 1924).

11. "Number Stories of Long Ago," stories which might be dramatized. (95).

12. Lantern Slides (92).

MATHEMATICAL GAMES

I shall attempt to describe only one of the many games suitable for mathematics clubs.

1. "Nim, a Game with a Complete Mathematical Theory." *The game here described has interested the writer on account of its seeming complexity, and its extremely simple and complete mathematical theory. The writer has not been able to discover much concerning its history, although certain forms of it seem to be played at a number of American colleges, and at some American fairs.*[1]

There are two players. They attempt to leave a set of three piles of counters in such a condition that the next player cannot take up the last counter. There is no arbitrary number of counters in each pile except that they shall not be equal at the beginning of the play. The first player selects one of the piles and removes as many counters as he chooses. He may take any number from one to all. The counters, however, must be taken from the same pile; and at least one must be taken. The player who takes up the last counter or counters wins.

The theory of the game is based on the possibilities for making a safe combination. Here are the theorems:

Theorem 1. If A leaves a safe combination on the table, B cannot leave a safe combination on the table at his next move.

[1] Charles Bouton, in: Jerbert, A.; A Photostat Scrap Book; University of Washington; unpaged.

THEOREM 2. If A leaves a safe combination on the table and B diminishes one of the piles, A can diminish one of the two remaining piles and leave a safe combination.[1]

2. Other games (7, 29, 43, 96).

3. THE MATHEMATICS TEACHER (May 1927, pp. 274-279); (Nov. 1924, pp. 422-425).

FIELD WORK AND FIELD TRIPS

1. Plane table measurements (81, pp. 280-282).

2. Simple measurements based on congruence of triangles (81, pp. 89-91).

3. Use of the transit.

4. A sight seeing trip. A trip might be planned with the object of recording mentally all of the examples seen of the triangle, or the rectangle, or the cylinder, or pyramid, or any other specific geometric figure. Groups or individuals might compete to see which one could record on paper the greatest number of examples found, when the members return to the club room.

OUTSIDE SPEAKERS

Some of the sources of outside speakers are listed:

1. Members of the faculty from the mathematics department.
2. Members of the faculty from the science department.
3. Members of the faculty from the mechanical drawing department.
4. Members of the faculty from the commercial department.
5. Local engineers.
6. Local photographer, mathematics in photography.
7. Local lawyer, mathematical thinking in law.

TOPICS FOR GROUP OR INDIVIDUAL PRESENTATION IN PLANE GEOMETRY

"Let no one ignorant of geometry enter my door."—Plato.

1. The ancient geometers, Thales, Pythagoras, Euclid, etc. (3; 11; 13; 35; 39; 41; 46; 87, p. 227).

2. Euclid's Elements (17; 48; 13; 87).

3. The "Pons Asinorum" (48, p. 251, Vol. I; 87, p. 270; 81, pp. 40 & 79).

[1] Essay by Charles Bouton in: Jerbert A.; Photostat Scrap Book; University of Washington, unpaged.

4. The three famous problems of antiquity (7, p. 337; 78, Vol. II, p. 10; 87, p. 256; 103, p. 297 Vol. II)—(1) Trisecting an angle (7, p. 344; 47, p. 59; 69, p. 49); (2) Squaring the circle (7, p. 346; 78, p. 329, Vol. I; 110, p. 126; 90, p. 112); (3) Duplicating the cube (7, p. 338; 78, p. 349; 47, p. 76).

5. A world of two dimensions only (1).

6. The fourth dimension (1; 74; 90).

7. Applications from astronomy (46; 5, p. 176).

8. The nine point circle (2 , p. 93; 110, p. 133).

9. The Simson Line (2, p. 115).

10. Homothetic figures (2, p. 32).

11. The Pythagoreans (13, p. 29; 46, p. 141; 87, p. 7).

12. The Pythagorean Theorem (32, p. 38; 37, p. 64; 47, p. 15; 48, p. 349; 87, p. 271).

13. Geometric representation of the irrational number; The "Unutterable"—(25, p. 99; 48, p. 35, Vol. I; 87, p. 183).

14. The Pythagorean numbers (87, p. 329; 25, p. 53).

15. Pascal's mystical hexagon.

16. Zeno's paradoxes (87, p. 308; 7, p. 84; 46, p. 271; 25, p. 122).

17. Non-Euclidean plane geometry (13, p. 266; 106; 110, p. 105; 113, p. 14; 87, p. 276).

18. Some different ways to make the fundamental constructions (49, p. 22; 52, p. 110; 37).

19. Fallacies and puzzles (7, p. 44; 7, p. 80; 7, p. 188; 110, p. 109).

20. The method of limits (110, p. 152; 87, p. 320).

TOPICS FOR GROUP OR INDIVIDUAL PRESENTATION IN THE FIELDS OF SOLID GEOMETRY AND TRIGONOMETRY

Geometria,
Through which a man hath the sleight
of length, and brede, of depth, of height.

—JOHN GOWER.

1. Archimedes and his contributions to solid geometry (87, p. 232; 41, p. 221; 47, p. 100).

2. Euclid's work in solid geometry (48, p. 24).

3. The Egyptians and the Great Pyramid.

4. Conic Sections (87, p. 281).

5. Perspective and projection (93, p. 269; 103, p. 338, Vol. II).
6. Trigonometry in animal mechanics (33, p. 46).
7. "Apparatus to illustrate line values of trigonometric functions" (110, p. 146).
8. John Napier (87, pp. 191 & 339; 110, p. 69).
9. History of the development of trigonometry (87, p. 291).

TOPICS FOR GROUP OR INDIVIDUAL PRESENTATION IN THE FIELD OF ALGEBRA

Algebra is generous, she often gives more than is asked of her.
—D'ALEMBERT.

1. Origin and use of x for the unknown (25, p. 85; 45, p. 57).
2. Nature and uses of formulae (25, p. 86).
3. The Binomial formula (25, p. 83; 87, p. 179; 103, pp. 507 & 511, Vol. II).
4. The quadratic equation and various methods of solution (87, p. 165; *School Science and Mathematics*, Nov. 1929).
5. Algebraic fallacies (110, p. 83).
6. Symbolism in algebra (25, p. 76; 87, p. 147; 103, p. 395, Vol. II).
7. Puzzles and fallacies (7; 29; 30; 110, p. 83).

TOPICS FOR GROUP OR INDIVIDUAL PRESENTATION IN THE FIELD OF NUMBER AND ARITHMETIC

There is no prophet which preaches the superpersonal God more plainly than mathematics.—PAUL CARSUS.

1. Origin and development of number notation (13, p. 13; 90, p. 1; 95; 87, p. 76).
2. The history of zero (87; 13, p. 14; 25, p. 19).
3. The history of the decimal point (110, p. 59).
4. The number system and the base 10 (35, p. 6; 25, p. 12; 110, p. 17).
5. The binary, and other number systems (35, p. 7; 87, p. 78).
6. A number system with base 12 (25; 31, p. 15; 87, p. 78).
7. The abacus (25, 13, p. 27; 103, p. 156, Vol. II; 87, p. 87).
8. Number sense in man and beast (25, p. 1; 35, p. 6).
9. Finger counting (25, p. 9; 35, p. 7; 156, p. 196, Vol. II; 87, p. 76).

10. The English tally stick (25, p. 23), (87, p. 26).
11. Reckoning in the middle ages (25, p. 26; 87).
12. Numeralogy (25, p. 38; 110, p. 180).
13. Amicable numbers (25, p. 44; 103, p. 23, Vol. II; 87, p. 74).
14. Perfect numbers (25, p. 44; 87, p. 74; 7, p. 37).
15. Prime numbers (25, p. 46; 7, p. 37; 69, p. 140; 7, p. 37; 87).

MISCELLANEOUS TOPICS FOR GROUP OR INDIVIDUAL PRESENTATION

If a man's wit be wandering, let him study mathematics; for in demonstrations, if his wit be called away ever so little, he must begin again.—Lord Bacon.

1. Origin and development of mathematical signs and symbols (35; 13, pp. 6-9; 87).
2. Mathematical prodigies (7, p. 264; 78, p. 219).
3. Graphic records and methods (69, p. 76).
4. Famous mathematicians of ancient times (Any History of Mathematics).
5. Famous mathematicians of modern times (See 1 under geometry).
6. Contemporary mathematicians.
7. Women mathematicians.
8. Highest possible degree of accuracy (110, p. 43).
9. Magic squares (90, p. 39; 110, p. 183; 3; 20; 65; 7, p. 137; 103, p. 591, Vol. II).
10. Simpson rule (32, p. 74).

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101. Smith, D. E.; *Teaching of Elementary Mathematics;* Macmillan.

102. Smith, D. E.; *Teaching of Geometry;* Ginn & Co.

103. Smith, D. E.; *History of Mathematics;* Ginn & Co.

104. Smith & Karpinski; *The Hindu-Arabic Numerals;* Ginn & Co.

105. Smith & Karpinski; *History of Japanese Mathematics;* Open Court Pub. Co.

106. Sommerville; *Non-Euclidean Geometry;* G. Bell & Sons, London.

107. Sykes; *Source Book of Problems for Geometry;* Allyn & Bacon.

108. Todhunter; *Elements of Euclid;* Macmillan.

109. Weeks; *A Boy's Own Arithmetic;* Dutton & Co.

110. White; *Scrap Book of Elementary Mathematics;* Open Court.

111. Whitehead, A. N.; *Introduction to Mathematics;* Henry Holt.

112. Woodring & Sanford; *Enriched Teaching of Mathematics in High School;* Teachers College, Columbia University.

113. Young, J. W. A.; *Lectures on Fundamental Concepts of Algebra and Geometry;* Macmillan.

114. Young, J. W. A.; *Monographs on Topics of Modern Mathematics;* Longmans.

On to Washington!

The next annual meeting of The National Council of Teachers of Mathematics will be held at the Raleigh Hotel in Washington, D.C., on Friday and Saturday, February 19 and 20, 1932. Members of the council who plan to attend the meeting should write at once to the Raleigh Hotel for reservations.

A tentative program for the meeting will be published in the January number of THE MATHEMATICS TEACHER, and the final program will appear in the February issue.

This meeting is sure to be one of the biggest and best meetings we have ever had. Plan to attend the meeting and urge other members to be present.

Problems of Algebra Pupils

By CLARA D. MURPHY

Evanston, Twp. High School, Evanston, Illinois

THIS TOPIC was chosen because I have the feeling that algebra teachers in general are not as conscious as they should be of the problems that confront their pupils. As they teach the same subject matter year after year, its difficulties become less apparent to them and the tendency is to forget that the ground is all new and more or less rough for their pupils. Before I began to teach, Professor Burrell of Wellesley College gave me this advice, "Always keep on hand a piece of work that is so difficult for you that you have sympathy for your pupils and understanding of the difficulties which they are encountering." I have not always followed this advice, but I believe in it.

Is it not also true that algebra teachers generally are too ready to attribute the shortcomings and the difficulties of their pupils to lack of thorough teaching in the previous years? I began my teaching with college students and I could not understand how they could come to college mathematics with so little comprehension of the underlying principles of the skills they were supposed to have acquired in elementary algebra. I promised myself that when I taught elementary algebra my pupils should learn these underlying principles and also go forth with some power to attack problems and carry them through to a successful solution. After several years, the opportunity came to work with high school pupils and again I could not understand how they could come up to high school with so little comprehension of fractions and so little power to read and get the meaning out of the printed page.

In a year or two I was engaged in a task that turned my thought from the trials of the algebra teacher to the problems of the algebra pupil, a task that was so difficult for me that I began to have sympathy for pupils in algebra and appreciation of their difficulties. This task was to reduce the number of failures in first-year algebra and at the same time to keep up the standard of work so that the majority of the pupils were prepared for a semester course which in turn prepared for college. As I studied the problem of the first-year algebra pupil, I became more and more convinced that it was not so much an

algebra problem as it was a first-year pupil problem, since in general the pupil who has difficulty with algebra is also in trouble with English, civics, and science—the other required subjects.

We have had some success in reducing our failure in algebra by modifying our courses. Segregating the poorest pupils, the use of a textbook that is written in language that the pupil can understand, and classroom procedure designed to take care of individual differences have helped in the first year. In the second year of algebra, the situation has been improved by making two courses—a year course for the average pupil and a semester course for the superior pupil, both giving the same credit. We still have failures, however, too many of them, and I believe they are largely due to the fact that the pupils do not know how to study.

I have come to this conclusion after frequent visits in the grade schools, numerous interviews with parents of failing pupils, and many attempts to help pupils analyze their own difficulties. When asked for the causes of failure, grade school teachers, parents, and pupils commonly say that the pupil knows how to do the work, but he makes mistakes; he does not know how to concentrate; or he is not interested in mathematics. Teachers of algebra generally add: the pupil is lacking in ability to read and he is not willing to work. As I see it, these are all the results of poor study habits, and I believe we can help the pupil in these difficulties if we can teach him how to study.

We have been trying for several years not only in algebra classes, but throughout the school to help our pupils acquire better habits of study. I propose to tell you what we are doing along this line, but I cannot tell you how successful our attempts have been because the whole project is in an experimental stage. We have, in our school, nine large home rooms in one of which each pupil has his own work desk where he may study before and after school hours and is required to study during the two periods when he is not in classes. The home room is kept quiet and everything is done to promote serious work on the part of the pupil. During the thirty-minute opening period, the home room director talks frequently about the connection between poor work and inefficient methods of study, and makes definite suggestions about acquiring good habits of study.

Last year we gathered together some suggestions for study in a pamphlet which was used with first-year pupils. I shall summarize part of them. You will notice that much of the material is borrowed

from current books on study. "The pole vaulter who fails knows that he has knocked off a bar; the football player knows when he misses a tackle. They want to know also why they failed, what is the difference between their technique and that of the ones who succeed?" In the same way the pupil who fails or does mediocre work should also want to learn about the methods used by the pupil who does excellent work.

At the start he should investigate and take care for the physical conditions which influence study—health, sleep, food, exercise, surroundings, and frame of mind when studying. He should find out when his best periods for work occur and make use of them. When he is feeling fresh, he should begin with his most difficult subject. When he is out of sorts, he should begin on something easy and pleasant.

This leads to the making of a study schedule which is an effective device for improving study habits. The pupil should include in it a definite time for the study of each subject each school day, a daily recreation hour, and an adequate period for week-end study. There is a printed sample for his use and his adviser is ready to help him plan the schedule and revise it when necessary.

Before sitting down for study he should have his tools ready for use—paper, pencils, textbook, assignment book, etc. He should be familiar with the study aids found in his textbook which are placed there to make his work easier and more interesting. He should write up his assignment book in an orderly way so that it will tell him the pages in the textbook, the points emphasized in the previous lesson, special things to do in attacking the new lesson, and the character and form of the written work required.

The pupil should begin work as soon as he sits down with the determination to work as hard as he can and as carefully as he can. He should start with a review of the preceding work whether he feels the need of it or not. Others know that this pays. He should keep working at problems that are hard for him because persistence will always win. When he makes a mistake, he should go back to fundamentals and substitute right thinking for the process used. In short, he should cultivate the habit of independent review. He should also cultivate the habit of independence in all of his work and take pride in thinking for himself.

Although these suggestions are offered for the study of all subjects, they are particularly adapted to the study of algebra. As soon

as the algebra pupil becomes interested in improving his technique, he begins to make progress. There are various ways of arousing his interest, depending on the individual. Some pupils like to know the reasons for studying algebra, for learning to solve equations, for making graphs, etc. Their interest is stimulated by a vision ahead and a glimpse of the relative values of the skills they are acquiring. Other pupils respond to the discussion of methods of work with their friends and teachers. Still others are stimulated by friendly rivalry into improving their technique. In short, their learning exercises have to be motivated.

We have a period after school every day for helping individuals and at that time we can set up goals a short distance apart for the encouragement of the weak. The momentum gained by crossing one goal line carries the pupil a long distance toward the next one. Crossing the goal line gives him real pleasure, and the feeling of success in mastering one unit of work makes him willing to tackle the next unit. Setting up goals of accomplishment is helpful to any pupil. I recall a boy in one of my classes who did good work but said that he had to spend two hours on his algebra every day. I persuaded him to set up time limits for his exercises and problems and to try to beat his previous record each day. He was dubious about his preparation and his marks under such conditions until I promised to consider the result of forty-five minutes' work satisfactory, no matter how small the quantity. In a week's time, he was doing all his required work in forty-five minutes and said he enjoyed his algebra more than before.

Last year several of our teachers helped pupils to solve their difficulties in algebra by giving them a list of questions regarding possible weaknesses. The pupils studied these and talked them over with their parents and the teacher, and in many cases were able to analyze their own difficulties. Some of these questions follow: Is the quality of the work that I am doing now the quality that I always intend to do? Do I realize that success is nine parts hard work and only one part brightness? To what is my failure due: Lack of interest? Lack of initiative? Lack of understanding of what to do? Superficial preparation? Poor reading ability? Poor memory? Guessing? Inability to apply myself completely and continuously? No independence? No desire to meet required standards? What are my weak points in mathematics? Common fractions? Decimals? Signed numbers? Equa-

tions? Translation into algebraic symbols? Analyzing word problems? Accuracy? etc. After the pupil and the teacher agreed on the places to be strengthened and the specific goal was set up, the pupil set to work cheerfully and gained ground.

A problem that bothers some algebra pupils is that during the class period they feel they understand perfectly the work that is being done, but when they continue the work alone at home, they are all in a muddle. To these pupils we suggest that they study their algebra as soon as possible after the class period because this gives maximum value from class instruction, and a brief review immediately, serves to fix important points. With this in mind we often use about a third of the class period for individual work on the next day's assignment.

Since one of the main purposes of home work is to "make the pupil proficient through practice in the knowledge and use of the subject matter presented in class," the pupil should feel a sense of mastery when he has done his home work. If he does not feel this sense of mastery, he should try to find the cause. Was he inattentive in class? Did he devote sufficient time to his work? Couldn't he concentrate because of distractions? Was the work too difficult? Did he copy the work from some other pupil? If the answer to the last question is yes, he might with profit think over this paragraph, quoting again from a pamphlet issued by one of the New York City schools.

"Your teacher determines from the home work submitted the weaknesses of his pupils and directs his teaching to help them in their difficulties. By temporarily deceiving him, are you helping him to assist you? Could copied home work give you the pleasure that comes from personal achievement? Does it fix in your mind the work of the previous day and lead you to understand better the work of tomorrow? Is it helping you to grow in your subject? Are you acquiring a sense of confidence in your ability to master the subject?"

I agree with Stone and Mallory when they say that a pupil who has a mathematical mind is merely a pupil who understands and masters each step of the mathematics developed day by day It is only by such an understanding of each single element and by a clear conception of how these elements contribute to the unity of the whole that a pupil obtains the valuable training which the study of mathematics can give. The algebra pupil must also learn the value of precision, the part that memorizing plays in success, the importance of

comprehending what he reads, and the art of analyzing a complex situation. As Morrison has said, "Algebra is primarily a method of thinking." "It does not enlarge the pupil's stock of things to think about, but it gives him the habit of expressing his thoughts in quantitative terms."

Learning the value of precision is a difficult problem for the algebra pupil. It takes time and effort to develop an attitude that demands absolute accuracy. We can tell him that proper arrangement of work and neat, legible writing are essential to accuracy in mathematics. We can advise him to examine his work to see that the steps in each exercise are arranged in an orderly way so that they are easy to read and understand. We can point out to him that the careful student develops the habit of scanning his work at frequent intervals, searching for possible mistakes. We can show him the importance of doing his work correctly the first time. But he must be desirous of becoming an accurate thinker and an accurate worker, and he must desire it enough to make real mental effort when doing work which is more or less mechanical.

Mechanical exercises present a problem to two types of pupil. One kind sails through a list in which the exercises are all of the same type, but does not recognize one of them in another environment. He can help himself, if he will take the time to find out why he is doing the exercises, to get a clear understanding of the principles involved, and to think about each exercise as he does it, instead of rushing to get through the list as soon as possible. Another pupil finds such work uninteresting and irksome. He too should think about the real need for acquiring the skill involved and the goal toward which he is working. He should realize that skill comes only with practice, but it will, when acquired, leave his mind free to carry on more interesting projects. Such a pupil can usually be interested in the mathematical vocabulary involved and in special devices for shortening the work. Making a game out of mechanical exercises will help both groups in speed, accuracy, and enjoyment of their work.

Everyone agrees that it is of little value to the pupil to learn to perform algebraic operations without a thorough understanding of the underlying principles, that it is also futile to follow blindly the directions of a memorized rule or to solve a problem automatically by fitting it into a stereotyped form. Yet the memory does play an important part in the success of the algebra pupil. It is necessary that

he be able to recall previously mastered skills and this recall is easier if he has at his command certain definitions, principles, and rules. When a pupil understands a unit of work, he should formulate and memorize the definitions and principles belonging to this unit so that they are a part of his mental equipment for future work. Unless he has been taught to do this, he is continually confronted by difficulties which should have been overcome in his previous mathematical experience.

Since memorizing seems to be so irksome and hard for the high school pupil these days, perhaps time might be well spent in giving the algebra pupil a few simple suggestions about the use of his memory power. He might be told that "the acquisition of knowledge involves the impression or learning of facts, their retention, the ability to recall them, and "the recognition of these facts as familiar when they are reproduced for effective use." He might be shown that "remembrance is proportional to the clearness with which he comprehends the subject matter," so that he must understand what he attempts to memorize. In other words, "he should make the first impressions carefully and slowly and clearly. Then he should repeat the impressions over a long period, repeating more frequently at the beginning and interspersing periods of impression with periods of recall. Memory operates according to these laws and when their conditions have been fulfilled, the memory can be trusted to retain the subject matter." Therefore, an attitude of confidence on the part of the pupil is necessary. If algebra teachers in planning the work, and algebra pupils in doing it, would take more account of the principles of memorizing, perhaps pupils would be better prepared for future mathematics courses.

Probably the word problem constitutes the greatest difficulty of the algebra pupil. In many cases this is due to the fact that he does not comprehend what he reads. The remedy for this lies in helping him with his reading. We hope in time to have a teacher of remedial reading. In the meantime some help can be given by the co-operation of the algebra and the English teachers in their common problem. Two of our teachers did this last year with success. A study of the pupil's standardized reading and grammar tests, a study of his algebra problems, and visiting each other's classes resulted in their being able to understand the pupil's difficulties in both subjects and in their being able to help him in overcoming them.

In the algebra class itself we have tried to make clear to the pupil the close connection between the reading of the problem and its solution by giving him in printed form the following directions and discussing them with him repeatedly. Read the problem carefully several times. Analyze the sentences—subject, predicate, etc. Look up the meanings of the doubtful words. Try to visualize the situation. Draw a diagram if possible. Have your facts clearly in mind before you begin to write because it is better to make the correct beginning the first time than to be obliged to make several beginnings. Think over the types of problems that you have previously learned to solve and see if your problem belongs to any one of these types.

What is the question of the problem? Let x represent the number of units in the answer. If more than one quantity is involved, represent the basic one by x and the others in terms of x, using the statements of relation in the problem. Many times the process of forming the equation is made easier by arranging the quantities in tabular form. Very often there is a sentence in the problem from which the equation can be made. If not, make a sentence which describes the relationship. The equation in algebra corresponds to the sentence in English. Find the subject and the predicate and then translate each part into algebraic symbols, using the expressions that you have already made. Reread the English sentence. Does the algebraic equation express the same relationship? After solving your equation, examine your results to see that they are reasonable and then check them in every statement of your problem, making good English sentences.

The word problem also constitutes the greatest difficulty of many algebra pupils, because it requires them to analyze a complex situation and see functional relationships. These are, of course, the most important of all the mental processes, since they are involved in any constructive work that the pupil may have to do. All reasoning or thinking has to be done by solving problems. The pupil must train himself to understand his problem clearly, to analyze it into its known and unknown elements, to recall in a systematic way similar situations and facts that bear directly on the problem, to decide on a method of solution, to carry it through to a logical conclusion, and to test his results. Here again he must have a certain amount of confidence that he can arrive at the correct solution. "In the improvement of his reasoning ability, the pupil's task is largely one of habit formation." "With every act of reasoning he is making him-

self into a careless reasoner or an accurate reasoner, a clear thinker or a muddy thinker."

Meanwhile, as the teacher and the pupil have worked on these problems of accuracy, memory, reading, and reasoning, they may have solved the problems that loom so large in the eyes of parents —concentration and interest. Ability to concentrate will improve sometimes unconsciously and again its improvement requires effort. If a pupil is easily susceptible to distractions, he should attack his work more vigorously. By doing this distractions are not so likely to occur, but when they do, he is able to "snap out of it" with greater ease. "Working absent-mindedly is not merely a waste of time; it is a bad habit."

"The pupil should have a definite goal, not vague good intentions. He should not take the attitude of trying to improve his will power in general, but seek to improve it in one concrete situation at a time." If he finds it hard to concentrate on algebra, he should deliberately force himself to work harder at it until it means something to him. He should try to find out why others enjoy it. "A strong motive plus genuine interest are most helpful in concentration."

It is perfectly possible to develop a genuine interest in a subject by working at it. The pupil who is willing to exert himself along the lines suggested by a good teacher often discovers that a subject becomes interesting in spite of the fact that at first it was distasteful or forbidding. "There is a vast difference between the quality of study which is done with interest and that done without it. Under the former conditions, the student is a creator; under the latter, a drudge."

As I have said before, we can, at the present time, only *hope* that our efforts at teaching good study habits will be successful, since we know so little about how this is done. I think, however, it is safe to say that the most difficult problems that the algebra pupil encounters are not those inherent in the subject matter, but those that arise from his lack of knowledge of how to study. Therefore, we should divide our efforts between trying to teach algebra and trying to teach effective habits of study if more of our pupils are to gain in power to think algebraically.

Pierre de Fermat

1601-1665

THROUGHOUT HIS lifetime, Pierre de Fermat was associated with the country in and about Toulouse. He studied law in the local university and later became a member of the local parliament. His leisure time was given over to the study of mathematics and he made contributions to most if not to all of the branches of mathematics then known. Fermat's discoveries were preserved in his *Varia Opera Mathematica* edited by his son Samuel de Fermat and published at Toulouse in 1679, in his notes to Bachet's translation of Diophantus which were also published by the younger Fermat in 1670, and in his correspondence with other mathematicians among whom were Roberval and Pascal. It is difficult, accordingly, to give the precise dates of Fermat's various discoveries, but it is probable that his work on maxima and minima was done by 1629. Professor Cajori says of this that "He substituted $x + e$ for x in the given function of x and then equated to each other the successive values of the function and divided the equation by e. If e be taken 0, then the roots of this equation are the values of x, making the function a maximum or a minimum. . . . The main difference between it and the rule of the differential calculus is that it introduces the indefinite quantity e instead of the infinitely small dx." This method was attacked by Descartes and the dispute that followed involved many of the important French mathematicians.

The translations from Fermat's work given in the *Source Book in Mathematics* indicate the breadth of his interests: On the Equation $x^n + y^n = z^n$, On the So-Called Pell Equation, On Analytic Geometry, Fermat and Pascal on Probability, and On Maxima and Minima.

Fermat has been called "the father of the modern theory of numbers." His contributions in this field were limited for the most part to the statement of theorems without proof, but occasionally with notations such as "I have discovered an admirable proof of this but the margin is too narrow to hold it."

Fermat's methods of proof are outlined in a manuscript belonging to Huygens discovered in the University of Leyden about fifty years

ago.* But the question has been raised as to whether Fermat had actual proofs for all his theorems or whether certain ones were not based on clever intuition. For example, consider the equation $x^n + y^n = z^n$, where x, y, z, and n are integers. Diophantus (*c.* 275) had discussed the problem of finding a square equal to the sum of two squares. Fermat states that it is impossible to divide a cube into two cubes, or in general any power whatever into two powers of the same denomination above the second. Later investigations have shown the truth of Fermat's statement for certain values of n but although no contrary case has been found, the general theorem has not yet been established.

Fermat's connection with the theory of probability is described by Professor Walker as follows:**

> The Chevalier de Méré, a gambler whom Leibniz described as having unusual ability "even for the mathematics" appears to have proposed certain questions to Pascal. . . . Among these questions was the celebrated "Problem of the Points" concerning the division of stakes between two players who separate without completing their game. Pascal and Fermat exchanged numerous letters on this subject during the year 1654, and in the course of this correspondence they generalized the problem more and more, until, at its close, that which had first appeared as a mere source of perplexity to a gambler had been elevated to a mathematical concept of great import.

* This is quoted by Ball, *Short Account of the History of Mathematics*, 1915 ed., pp. 296-298.

** Helen M. Walker, *Studies in the History of Statistical Method*, Baltimore, 1929, p. 6.

NEWS NOTES

A New College in 1932

Teachers College, Columbia University, announces a plan for a new type of teacher training institution to open September, 1932. It will operate as an undergraduate unit at the college level.

This new college, to be under the direction of Dr. Thomas Alexander, professor of Education, Teachers College, will attempt to demonstrate radically different methods in the selection and training of young men and women who are to become teachers in nursery, elementary and secondary schools. While preparing these young people for teaching positions the new unit, which will grant the Bachelor of Science and the Master's degrees, will operate also as a demonstration college in which graduate students in Teachers College

may observe improved methods in teacher training.

In this respect the new college will be to the field of teacher training what the Lincoln and Horace Mann Schools of Teachers College are to the field of elementary and secondary education. As these schools attempt to create and demonstrate more effective procedures in elementary and secondary education, the new college will attempt to create and demonstrate improved procedures in the training of teachers for elementary, secondary and nursery schools.

Rigid methods of elimination will be used in the selection of students. High school and college executives throughout America will be asked to cooperate in selecting those who show the richest promise of developing into highly competent teachers and educational leaders. Selection will be on the basis of good health, sound scholarship, desirable personal qualities, and promise of unusual growth. The student body will not be restricted to local sections of the country but will be chosen from a wide geographical area.

This unusual care in creating the student body, to be limited the first year to 100 young men and 100 young women of outstanding ability and personality, will be taken because it is felt that mediocrity is today the curse of the teaching profession. Insistence that half the students be young men is based on the conviction that teaching, almost monopolized at present by women, should be a profession for men as well.

The duration of the period of study in the college will vary approximately from three to five years according to the ability of the student, and will include at least one year spent in study and travel abroad. Students will be required to spend some time in actual work in industry and business so that when they become teachers they will have an adequate conception of the work of the world into which most pupils who graduate from our American schools must enter. One year of satisfactory teaching service in co-operating private and public school systems will be required before a degree is granted. Assurance of such co-operation has been obtained from a number of school systems.

Cost of attendance at the college probably will not exceed $1,000 a year. It is hoped that a number of scholarships may become available for unusually promising young men and women who wish to prepare for the teaching profession and who cannot meet tuition costs.

If the college is successful in achieving its purpose, it itself will not only develop teachers far superior to most of those of the present day but its procedures will be adopted by teacher training institutions throughout the nation to bring an educational reconstruction that, first and last, depends upon a new race of teachers.

With its emphasis upon quality rather than quantity the new college, in so far as it succeeds and becomes influential in creating such a race of teachers, should tend to reduce the present over-supply of teachers of mediocre ability and personality and make the teaching profession as attractive to American youth of superior ability and personality as those professions and calling in which only the superior person can achieve success that is measured by social usefulness as well as by financial remuneration.

Students in the new college will at first be housed in the present dormitories of Teachers College and will utilize existing classrooms and laboratories. It

is hoped that eventually new buildings will be erected to house the college. Effort will be made to utilize to the utmost the opportunity which New York City provides as a rich cultural center. Moreover the social life of the students will comprise an important factor of their training program.

Because problems of the future teacher will center about the child, the central core of the curriculum will deal with child nature and development and the students in training will have intimate contact, under careful guidance, with the children in laboratory schools of Teachers College and other institutions.

Because the future teacher must in a real sense be a social worker, the curricula of the college will provide courses in social economy, sociology, economics, politics, and problems of civic and industrial life, and will look forward to providing each student active participation in some form of social work. An all-year camp is planned as an adjunct to the college to provide opportunities for field work in physical education, biology, astronomy, and other fields of science.

The college will reject the traditional point system. Graduation will be based upon satisfactory examinations of practical as well as academic character.

Faculty members of the new college will be selected as carefully as are the students. There will be close, intimate association between the student body and an outstanding faculty to bring about the contact of mind with mind, that contact of the spirit of the teacher with the ripening enthusiasm of the pupil, which is the most important of all factors in education.

The mathematics section of The Northeastern Ohio Educational Association met in Cleveland October 30. The following program was given:

Leader, Mary Dilley, Alliance.

1. Some points in classroom technique in algebra, Vera Sanford, School of Education, W. R. U.
2. What mathematics shall we teach? to whom? and why? Marie Gugle, Assistant Superintendent, Columbus.
3. Mathematics a tool—subject—and much more, W. D. Cairns, Dept. of Mathematics, Oberlin College.

Section 19, of the New York Society for the Experimental Study of Education, held its first dinner meeting of the year on Saturday, October 24, 1931, at the Men's Faculty Club of Columbia University when over one hundred were present to hear Professor Reeve, the chairman of the club, tell about his trip around the world. He, also, spoke on "The Teaching of Mathematics in Germany."

Index photographed at the beginning for the convenience of the microfilm user.

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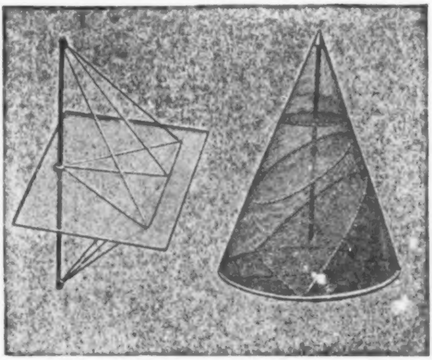

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